Reparametrization invariance and Hamilton-Jacobi formalism

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Abstract

Systems invariant under the reparametrization of time were treated as constrained systems within Hamilton-Jacobi formalism. After imposing the integrability conditions the time-dependent Schrödinger equation was obtained. Three examples are investigated in details.

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1 Introduction

Reparametrization theories of gravity such as general relativity and string theory are invariant under reparametrization of time [1, 2, 3, 4]. A transformation from a reparametrization-invariant system to an ordinary gauge system was applied for deparametrizing cosmological models [5]. 1By adding a surface term to the action functional the gauge invariance of the systems whose Hamilton-Jacobi equation is separable was improved [6]. Reparametrization invariance was treated as a gauge symmetry in [3] and a time-dependent Schrödinger equation for systems invariant under the reparametrization of time was developed [7, 8]. To reach this goal an additional invariance action was introduced without changing the equation of motions but modifying the set of constraints [7, 8]. After the second class constraints were eliminated one of first class constraints becomes time-depending Schrödinger equation.

An intriguing question arises: could we construct a time-depending Schrödinger equation without involving any gauge invariance transformations for the extended action proposed in [7, 8]?

The answer of this question is to use the Hamilton-Jacobi (HJ) formalism for constrained systems [9], based on Caratheodory's equivalent Lagrangians method [10]. This formalism does not differentiate between the first and second class constraints, we do not need any gauge fixing terms and the action provided by (HJ)

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is useful for the path integral quantization method of the constrained systems. In addition, it was proved that the integrability conditions of (HJ) formalism and Dirac's consistency conditions are equivalent [11] and the equivalence of the chain method [12] and (HJ) formalism was investigated [13].

The main aim of this paper is to obtain, using (HJ) formalism, the time-dependent Schrödinger equation for theories invariant under the time reparametrization.

The plan of the paper is as follows:
In Sec. 1 the (HJ) formalism is briefly presented. In Sec. 2 the reparametrization invariance is investigated with (HJ) approach. Sec.3 is dedicated to our conclusions.

2 Hamilton-Jacobi formalism

(HJ) formalism based on Caratheodory's equivalent Lagrangians method [10] is an alternative method [9] of quantization of constrained systems. The basic idea of this new approach is to consider the constraints as new "Hamiltonians" and to involve all of them in the process of finding the action. Let us consider that a given degenerate Lagrangian L admits the following primary "Hamiltonians"

$$H'_\alpha = p_\alpha + H_\alpha(t_\beta, q_a, p_a). \tag{1}$$

The canonical Hamiltonian H_0 is defined as

$$H_0 = -L(t, q_i, \dot{q_\nu}, \dot{q_a} = w_a) + p_a w_a + \dot{q_\mu} p_\mu; \quad \nu = 0, n-r+1, ..., n. \tag{2}$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad dp_\mu = -\frac{\partial H'_\alpha}{\partial t_\mu} dt_\alpha, \quad \mu = 1, ..., r. \tag{3}$$

and the (HJ) function is given by

$$dz = (-H_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_a}) dt_\alpha \tag{4}$$

The set of equations (3) is integrable if and only if [9]

$$[H'_\alpha, H'_\beta] = 0 \quad \forall \alpha, \beta. \tag{5}$$

The method is straightforward for constrained systems having finite degree of freedom [14] but it becomes in some cases quite difficult to be used for field theories. The main difficulty comes from the fact that some surface terms may play an important role in closing the algebra of "Hamiltonians" but some of them have no physical meaning from the (HJ) point of view. Another problem is the treatment of the second-class constraints systems within (HJ) formalism [15]. In this particular case the "Hamiltonians" are not in involution and it is not a unique way to solve this problem [16, 17, 18].

3 Reparametrization invariance

First of all we mention that if we apply naively the (HJ) formalism to reparametrization invariance theories no time-dependent Schrödinger equation appears simply because of the equivalence of (HJ) and Dirac's formalisms.

In order to obtain a time-dependent Schrödinger equation the initial Lagrangian is extended with a term involving the lapse function N and in the last example with its corresponding superpartners. In this manner a second class-constrained system is obtained. This system is investigated within (HJ) formalism and after imposing the integrability conditions we obtained the time-dependent Schrödinger equation on the surface of constraints.

3.1 Examples

3.1.1 Non-relativistic particle dynamics

The Lagrangian for a non-relativistic particle moving in the three dimensional space is given by

$$L = \frac{1}{2} m {\dot{x}_i}^2(t) - V(x_i), \tag{6}$$

where $x_i, i = 1, 2, 3$ are dynamical variables, t denotes the ordinary physical time parameter, m is the mass of the particle and $V(x_i)$ is the potential.

Treating the time as a dynamical quantity the Lagrangian (6) takes the form

$$L^{'} = \frac{1}{2N(\tau)} m {\dot{x}_i}^2(t) - N(\tau) V(x_i(\tau)) \tag{7}$$

where $\dot{x}_i = \frac{dx_i}{d\tau}$ and $N(\tau)$ is the lapse function relating the physical time to the arbitrary parameter by $dt = N(\tau) d\tau$.

In order to obtain a time-dependent Schrödinger equation we consider the extended Lagrangian

$$L^{''} = \frac{1}{2N(\tau)} m {\dot{x}_i}^2(t) - N(\tau) V(x_i(\tau)) + \lambda(-N + \frac{dt}{d\tau}) \tag{8}$$

From (8) we obtain

$$p_i = m \frac{\dot{x}_i}{N}, p_\lambda = 0, p_N = 0, p_t = \lambda. \tag{9}$$

Using (8) and (9) the expression for canonical Hamiltonian becomes

$$H_c = N(\frac{{p_i}^2}{2m} - V(x_i) + \lambda) + \dot{t}(p_t - \lambda) \tag{10}$$

Therefore in (HJ) formalism we have the following "Hamiltonians":

$$H_0^{'} = p_0 + N(\frac{{p_i}^2}{2m} - V(x_i) + \lambda), H_1^{'} = p_\lambda, H_2^{'} = p_N, H_3^{'} = p_t - \lambda \tag{11}$$

We observe from (11) that the system of second class constraints in Dirac's classification and H_c is a combination of $H_0 = \frac{{p_i}^2}{2m} - V(x_i)$, λ and $H_3^{'}$. To make the system integrable we impose the integrability conditions. Making the variation of $H_2^{'}$ zero we obtained another "Hamiltonian"

$$H_4^{'} = \lambda + \frac{{p_i}^2}{2m} - V(x_i). \tag{12}$$

On the surface of constraints it is easy to realize that $p_t = \lambda$ and (12) represents the time-dependent Schrödinger equation.

3.2 Relativistic point particle

The well known Lagrangian for a free relativistic particle is given by

$$L = -m\sqrt{1 - \dot{x}_i^2}. \tag{13}$$

Here m, t and x_i are, the mass, proper time and the position of the particle. If we make the reparametrization transformation

$$dx^0 = N(\tau) d\tau, \tag{14}$$

from (13) it follows that

$$L^{'} = -m\sqrt{N^2(\tau) - \dot{x}_i^2}. \tag{15}$$

To obtain a time-dependent Schrödinger equation we consider the extension of (15) as

$$L^{''} = -m\sqrt{N^2(\tau) - \dot{x}_i^2} + \lambda(\dot{x}^0 - N) \tag{16}$$

From (16) we obtain

$$p_i = m\frac{\dot{x}_i}{\sqrt{N^2 - \dot{x}_i^2}}, p_\lambda = 0, p_N = 0, p_{x^0} = \lambda \tag{17}$$

Taking into account (16) and (17), the corresponding canonical Hamiltonian is

$$H_c = N(\sqrt{p_i^2 + m^2} + \lambda) + \dot{x}^0(p_{x^0} - \lambda). \tag{18}$$

In this case the "Hamiltonians" of (HJ) formalism are given by

$$H_0^{'} = p_0 + N(\sqrt{p_i^2 + m^2} + \lambda), H_1^{'} = p_\lambda, H_2^{'} = p_N, H_3^{'} = p_{x^0} - \lambda \tag{19}$$

Since the set from (18) and (19) is not integrable in (HJ) formalism, we used the same procedure as in the previous example and on the surface of constrains, the time-dependent Schrödinger equation is obtained as

$$H_4^{'} = p_{x^0} + \sqrt{p_i^2 + m^2}, \tag{20}$$

which represents our desired result.

3.3 N=2, D=1 supersymmetry

In [8] the N=2 supersymmetric quantum mechanics was coupled to world-line supergravity and the local supersymmetric action was obtained. The action to start with is

$$S = S_{1(n=2)} + S_{2(n=2)}, \tag{21}$$

where

$$S_{1(n=2)} = \int \{ \frac{(Dx)^2}{2N} - i\bar{\chi}D\chi - 2N(\frac{\partial g}{\partial x})^2 - 2N\bar{\chi}\chi\frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial x}(\bar{\psi}\chi - \psi\bar{\chi})\}, \tag{22}$$

$$S_{2(n=2)} = \int \{\lambda(-N + \dot{t} + \frac{\bar{\psi}}{2}\bar{\eta} - \frac{\psi}{2}\eta) - \lambda_1(i\dot{\eta} + \frac{V}{2}\eta + \frac{\bar{\psi}}{2}) + \lambda_2(-i\dot{\bar{\eta}} + \frac{\bar{\psi}}{2} + \frac{V}{2}\bar{\eta})\} \tag{23}$$

Here the covariant derivatives are given by $Dx = \dot{x} - \frac{i}{2}(\psi\bar{\chi} + \bar{\psi}\chi)$ and $D\chi = \dot{\chi} + \frac{i}{2}V\chi$ respectively. λ_1 and λ_2 are the superpartners of λ and ψ,$\bar{\psi}$ and V are superpartners of N.For more details concerning the above action see [8] and the references therein. We observed that in addition to the primary "Hamiltonians"

$$H_1^{'} = p_N, H_2^{'} = p_\psi, H_3^{'} = p_{\bar{\psi}}, H_4^{'} = p_V, \tag{24}$$

from (22) and (23) we obtained a new set of "Hamiltonians" having the following forms

$$H_5^{'} = p_\lambda, H_6^{'} = p_t - \lambda, H_7^{'} = p_{\lambda_1}, H_8^{'} = p_{\lambda_2}, H_9^{'} = p_\eta + i\lambda_1, H_{10}^{'} = p_{\bar{\eta}} + i\lambda_2. \tag{25}$$

By inspection we observed that the set of "Hamiltonians" from (25) is not in involution, so the system of total differential equations corresponding to (24) and (25) is not integrable within (HJ) formalism. The canonical Hamiltonian becomes

$$H_c = N(\lambda + H_0) - \frac{\psi}{2}(S_{\bar{\eta}} + \bar{S}) + \bar{\psi}(-S_\eta + S) + \frac{V}{2}(F_\eta + F), \tag{26}$$

where

$$H_0 \ = \frac{p^2}{2} + 2(\frac{\partial g}{\partial x})^2 + 2(\frac{\partial^2 g}{\partial x^2})\bar{\chi}\chi, S_\eta = (\bar{\eta}p_t - p_\eta), S_{\bar{\eta}} = (p_{\bar{\eta}} - \eta p_t),$$

$$F_\eta = (\eta p_\eta - \bar{\eta} p_{\bar{\eta}}), F = \bar{\chi}\chi, S = (ip + 2\frac{\partial g}{\partial x})\chi, \bar{S} = (-ip + 2\frac{\partial g}{\partial x})\bar{\chi} \quad (27)$$

The "Hamiltonians" from (24) and (25) together with

$$H_0^{'} = p_0 + H_c \quad (28)$$

form the total set "Hamiltonians" for our investigated problem. The next step is to investigate the corresponding integrability conditions. Firstly we impose the variations of "Hamiltonians" given by (24) to be zero. As a result we obtained a new set of "Hamiltonians" as given below

$$H_0^{''} = \lambda + H_0, Q_\eta = -S_\eta + S, Q_{\bar{\eta}} = S_{\bar{\eta}} + \bar{S}, \tilde{F} = F_\eta + F \quad (29)$$

On the surface of constraints of (25) the set of "Hamiltonians"from (29) is in involution and we obtained $H_0^{''} = p_t + H_0$ which represents the time-dependent Schrödinger equation.

4 Conclusions

In this paper we proved that a time-dependent Schrödinger equation was obtained for a given reparametrization theory by using an extended Lagrangian involving the lapse function N and its superpartners. Taking into account that (HJ) and Dirac's formalisms are equivalent we observed that the reparametrization invariance theories are not integrable inside of the (HJ) formalism. This result is due to the fact that the integrability conditions (5) are not satisfied. On the surface of constraints we obtained the time-dependent Schrödinger equation without involving any symmetry transformation of the extended action.

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